5: Operads and their operator categories

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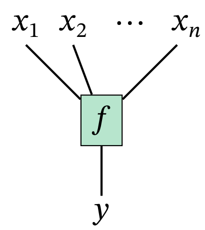
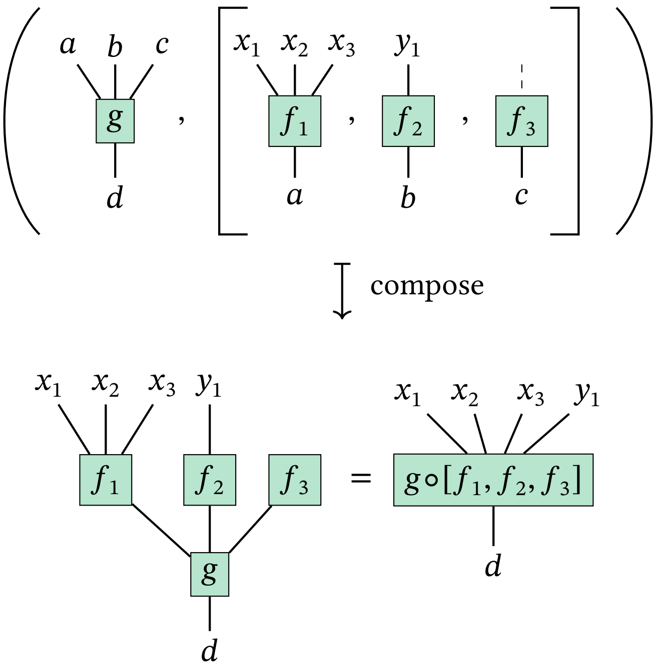
These notes roughly follow [Hau23, §2].

1 Classical operads

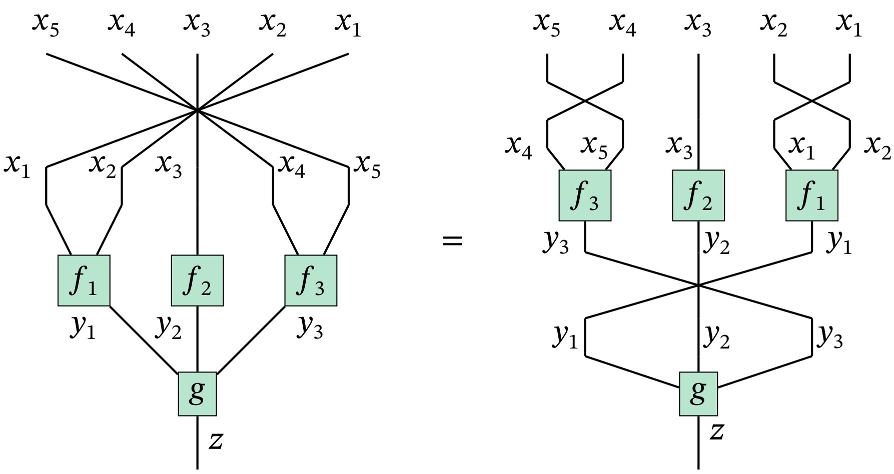
Idea: An operad provides a framework for composing operations of any finite arity. Abstractly, we can think of an n-ary operation f with inputs $x_1, x_2, \ldots, x_n$ and output y as a labeled diagram.



If the inputs and outputs of operations are compatible, they can be composed which corresponds to stacking their diagrams. We always a compose an n-ary operation with a list of n compatible operations – one for each input.



This composition should be associative, and for each label x we require an identity operation. In addition, this composition should be compatible with permutations of inputs. For example, we would like to have the following equality.



We will summarize this in the following (somewhat informal) definition.

Definition 1.1. An *operad* $\mathcal{O}$ consists of

- a set $\mathrm{Ob}\,\mathcal{O}$ of *objects*,
- for all lists of objects $X = (x_1, \dots, x_n)$ $(n \geq 0)$ and $y \in \mathrm{Ob}\,\mathcal{O}$ a set $\mathrm{Hom}_{\mathcal{O}}(X, y)$ of *multimorphisms* and for all lists of objects $Y = (y_1, \dots, y_n)$ $(n \geq 0)$ and $x \in \mathrm{Ob}\,\mathcal{O}$ a set $\mathrm{Hom}_{\mathcal{O}}(Y, x)$ of *multimorphisms*

- a composition operation that lets us compose a multimorphism $g\colon Y \longrightarrow z$ with a list of multimorphisms $F = (f_i\colon X^i \longrightarrow y_i)$ to get a multimorphism $g \circ F\colon (X^1, \dots, X^n) \longrightarrow z$
- an identity $\mathrm{id}_x\colon (x) \longrightarrow x$ for each object x
- a permutation symmetry whereby for each $\sigma \in S_n$ we have

$$\mathrm{Hom}_{\mathcal{O}}((x_1, \dots, x_n), y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}((x_{\sigma(1)}, \dots, x_{\sigma(n)}), y)$$

such that the composition is

- associative, i.e. $(f \circ G) \circ (H_1, \dots, H_n) = f \circ (G \circ (H_1, \dots, H_n))$ where $G = (g_1, \dots, g_n)$ and $G \circ (H_1, \dots, H_n) = (g_1 \circ H_1, \dots, g_n \circ H_n)$,
- unital, i.e. $\mathrm{id}_y \circ f = f = f \circ \mathrm{id}_X$ where $\mathrm{id}_X = (\mathrm{id}_{x_1}, \dots, \mathrm{id}_{x_n})$,
- compatible with the symmetric group actions (in a sense that is obvious but annoying to write down).

Example 1.2. The *commutative operad* Comm has

- a single object $*$, and
- for each arity $n = 0, 1, \dots$ a single multimorphism.

Example 1.3. The *associative operad* Assoc has

- a single object $*$, and
- $\mathrm{Hom}_{\mathsf{Assoc}}(*^n, *) = S_n$ with the natural S_n action. Composition is given by concatenation of block permutations.

Example 1.4. The operad CM has objects a, m and multimorphisms

$$\mathrm{Hom}_{\mathsf{CM}}(X, y) = \begin{cases} *, & \text{if } X = (a, \dots, a) \text{ and } y = a, \\ *, & \text{if } X = (a, \dots, a, m) \text{ up to permutation, and } y = m, \\ \varnothing, & \text{otherwise.} \end{cases}$$

Example 1.5. Every category C can be regarded as an operad with objects $\mathrm{Ob}(C)$ and only unary operators.

Example 1.6. Let $(\mathcal{V}, \otimes, 1)$ be a symmetric monoidal category. Define an operad $\mathcal{V}_{\mathrm{opd}}$ with

- the same objects as $\mathcal{V}$,
- $\mathrm{Hom}_{\mathcal{V}_{\mathrm{opd}}}(\emptyset, y) := \mathrm{Hom}_{\mathcal{V}}(1, y)$,
- $\mathrm{Hom}_{\mathcal{V}_{\mathrm{opd}}}((x_1, \dots, x_n), y) := \mathrm{Hom}_{\mathcal{V}}(x_1 \otimes \cdots \otimes x_n, y)$ (using a fixed convention for bracketing tensor factors),
- composition is given by re-associating, tensoring and composing in $\mathcal{V}$.

Definition 1.7. A functor of operads $F\colon \mathcal{O} \longrightarrow \mathcal{P}$ consists of

- A map of objects $F\colon \mathrm{Ob}(\mathcal{O}) \longrightarrow \mathrm{Ob}(\mathcal{P})$
- Maps of multimorphisms $\mathrm{Hom}_{\mathcal{O}}((x_1, \dots, x_n), y) \longrightarrow \mathrm{Hom}_{\mathcal{P}}((Fx_1, \dots, Fx_n), Fy)$ that are compatible with composition, identities and permutations.

Operads and their functors assemble into a category **Opd**. If $\mathcal{V}$ is a symmetric monoidal category, then a functor $\mathcal{O} \longrightarrow \mathcal{V}_{\mathrm{opd}}$ is called an $\mathcal{O}$-algebra in $\mathcal{V}$.

Example 1.8. Let $\mathcal{V}$ be a symmetric monoidal category.

1. A Comm-algebra in $\mathcal{V}$ is a commutative algebra: The functor picks out an object $A \in \mathcal{V}$ and morphisms $\mu_n\colon A^{\otimes n} \longrightarrow A$ satisfying the following properties.

   - Each μ_n is invariant under tensor factor permutations.
   - If $n = n_1 + \cdots + n_k$ is a partition of n, then
   $$\mu_n = \mu_k \circ (\mu_{n_1} \otimes \cdots \otimes \mu_{n_k}).$$

   The morphism $\mu_0\colon 1 \longrightarrow A$ serves as the unit and $\mu_2\colon A \otimes A \longrightarrow A$ as the multiplication. The equational laws follow from the properties above. For $n > 2$, we can recover μ_n by iterating μ_2.

2. An Assoc-algebra is an associative algebra: The functor picks out an object $A \in \mathcal{V}$ and $\mu_2\colon A \otimes A \longrightarrow A$. For each ordering of $\{1, \dots, n\}$ we specify a morphism $A^{\otimes n} \longrightarrow A$ but all of them are obtained from each other by permuting the tensor factors and iterating the binary operation.

3. A CM-algebra is a commutative algebra A plus a left A-module M: The functor maps $a \longmapsto A$ and $m \longmapsto M$.

Observation 1.9. Let $\mathcal{V}$ and $\mathcal{W}$ be symmetric monoidal categories. A functor of operads $\mathcal{V}_{\mathrm{opd}} \longrightarrow \mathcal{W}_{\mathrm{opd}}$ is the same as a lax-monoidal functor $\mathcal{V} \longrightarrow \mathcal{W}$, since every multimorphism factors uniquely as

$$(x_1, \dots, x_n) \longrightarrow x_1 \otimes \cdots \otimes x_n \longrightarrow y$$

so all the information is contained in the mapping of the unary operations together with the mappings of the form

$$(Fx_1, \dots, Fx_n) \longrightarrow F(x_1 \otimes \cdots \otimes x_n) \quad \text{in } \mathcal{W}_{\mathrm{opd}}$$

which correspond to $Fx_1 \otimes \cdots \otimes Fx_n \longrightarrow F(x_1 \otimes \cdots \otimes x_n)$ and provide the lax-monoidal structure.

Proposition 1.10. *An operad $\mathcal{O}$ arises from a symmetric monoidal category if and only if*

1. *For all $X = (x_1, \dots, x_n)$ in $\mathcal{O}$ there exists an object $\otimes X$ and multimorphism $\otimes_X \colon X \longrightarrow \otimes X$ such that composition with it gives a bijection*

$$\mathrm{Hom}_{\mathcal{O}}(X, y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(\otimes X, y)$$

   for all $y \in \mathcal{O}$.

2. *If $X = (X_1, \dots, X_n)$ as sublists, then*

$$\otimes_{(\otimes_{X_1}, \cdots \otimes_{X_n})} \circ (\otimes_{X_1}, \dots, \otimes_{X_n}) \colon X \longrightarrow \otimes(\otimes_{X_1}, \dots, \otimes_{X_n})$$

   corresponds to an isomorphism $\otimes X \longrightarrow \otimes(\otimes_{X_1}, \dots, \otimes_{X_n})$ in $\mathcal{O}$ under the identification above.

2 Categories of operators

Idea: We perform something similar to a "Grothendieck construction" to package all the data of an operad into a fibration which is a plain 1-categorical object. These categories will be fibered over finite pointed sets. This leads to an alternative definition of operad which readily lifts to the ∞-world. In particular this allows us to define symmetric monoidal ∞-categories.

Definition 2.1. Write $\mathbb{F}_*$ for the category of finite pointed sets of the form

$$\langle n \rangle := (\{0, 1, 2, \dots, n\}, 0)$$

and basepoint-preserving maps.

Remark 2.2. We can interpret a morphism $f\colon \langle n\rangle \longrightarrow \langle m\rangle$ in $\mathbb{F}_*$ as a partial function

$$\tilde{f}\colon \{1,\dots,n\} \longrightarrow \{1,\dots,m\}$$

where $\tilde{f}(x)$ is undefined, whenever f maps x to the basepoint (think "Maybe monad").

Remark 2.3. In $\mathbb{F}_*$ we have $\langle n\rangle \cong \prod_{i=1}^{n}\langle 1\rangle$ which is witnessed by the projections

$$\rho_i\colon \langle n\rangle \longrightarrow \langle 1\rangle, \quad x \longmapsto \begin{cases} 1, & \text{if } x = i \\ 0, & \text{otherwise.} \end{cases}$$

We call these projections *Segal maps.* (Compare to the "simplicial" Segal maps from the monoid talk!) To see that they are indeed projections, observe that every morphism $f\colon \langle m\rangle \longrightarrow \langle n\rangle$ is uniquely determined by the fibers $f^{-1}(1),\dots,f^{-1}(n)$ and that we have

$$f^{-1}(j) = (\rho_j \circ f)^{-1}(1).$$

Definition 2.4. Let $\mathcal{O}$ be an operad. Its *category of operators* $\mathcal{O}^{\otimes}$ has the following data.

- Objects are tuples $(x_1,\dots,x_n)$ with $x_i \in \operatorname{Ob}\mathcal{O}$ and $n \in \mathbb{N}_{\geq 0}$.
- A morphism $(x_1,\dots,x_n) \longrightarrow (y_1,\dots,y_m)$ in $\mathcal{O}^{\otimes}$ consists of
  - a morphism $f\colon \langle n\rangle \longrightarrow \langle m\rangle$ in $\mathbb{F}_*$ and
  - a family of multimorphisms $\Phi_j\colon (x_i)_{i\in f^{-1}(j)} \longrightarrow y_j$ in $\mathcal{O}$ for $j = 1,\dots,m$.
- Given morphisms $(x_1,\dots,x_n) \xrightarrow{(f,\Phi)} (y_1,\dots,y_m) \xrightarrow{(g,\Psi)} (z_1,\dots,z_k)$ their composite

  $$(x_1,\dots,x_n) \xrightarrow{(h,\Gamma)} (z_1,\dots,z_k)$$

  is given by
  - $h \coloneqq gf\colon \langle n\rangle \longrightarrow \langle k\rangle$ in $\mathbb{F}_*$
  - $\Gamma_j \coloneqq \Psi_j \circ (\Phi_i)_{i\in g^{-1}(j)}$ in $\mathcal{O}$ for $j = 1,\dots,k$.
- The identity on $(x_1,\dots,x_n)$ is given by $\mathrm{id}\colon \langle n\rangle \longrightarrow \langle n\rangle$ together with $\mathrm{id}_{x_1},\dots,\mathrm{id}_{x_n}$.

Remark 2.5. The category $\mathcal{O}^{\otimes}$ admits a canonical projection functor

$$p\colon \mathcal{O}^{\otimes} \longrightarrow \mathbb{F}_*, \quad \Big((x_1,\dots,x_n) \xrightarrow{(f,\Phi)} (y_1,\dots,y_m)\Big) \longmapsto f.$$

We will see that p is a certain "fibration" from which we can reconstruct $\mathcal{O}$.

Definition 2.6. A morphism $f\colon \langle n\rangle \longrightarrow \langle m\rangle$ in $\mathbb{F}_*$ is called

- *inert*, if $|f^{-1}(j)| = 1$ for all $j = 1, \dots, m$ (i.e. f is a bijection away from 0),
- *active*, if $|f^{-1}(0)| = 1$ (i.e. f is defined everywhere).

Example 2.7.

- There is a unique active map $\alpha_n\colon \langle n\rangle \longrightarrow \langle 1\rangle$. It sends $j \longmapsto 1$ for all $j \geq 1$.
- For $n \geq 2$ the Segal maps $\langle n\rangle \longrightarrow \langle 1\rangle$ inert but not active.
- Every automorphism $\langle n\rangle \longrightarrow \langle n\rangle$ is inert and active.

Proposition 2.8. *Every morphism $f\colon \langle n\rangle \longrightarrow \langle m\rangle$ in $\mathbb{F}_*$ factors uniquely up to iso as $f = hg$ with h active and g inert. (Active and inert maps form a factorization system).*

Definition 2.9. Let $p\colon \mathcal{E} \longrightarrow \mathcal{B}$ be a functor and $f\colon x \longrightarrow y$ in $\mathcal{B}$.

- A *lift* of f is a morphism $\overline{f}\colon \overline{x} \longrightarrow \overline{y}$ in $\mathcal{E}$ such that $p(\overline{f}) = f$.
- A lift $\overline{f}\colon \overline{x} \longrightarrow \overline{y}$ of f is called *cocartesian*, if it has the following universal property: for all $\phi\colon \overline{x} \longrightarrow \overline{z}$ in $\mathcal{E}$ and $g\colon y \longrightarrow z$ in $\mathcal{B}$ with $p(\phi) = gf$ there exists a unique lift $\overline{g}\colon \overline{y} \longrightarrow \overline{z}$ such that $\phi = \overline{g}\overline{f}$.

$$\begin{tikzcd}
 & & \overline{y} \arrow[dr, dashed, "\exists!\,\overline{g}"] & \\
\mathcal{E} \arrow[dd, "p"] & \overline{x} \arrow[ur, "\overline{f}"] \arrow[rr, bend right, "\phi"] & & \overline{z} \\
 & & y \arrow[dr, "g"] & \\
\mathcal{B} & x \arrow[ur, "f"] \arrow[rr, bend right, "p(\phi)"'] & & z
\end{tikzcd}$$

- We say that p has cocartesian lifts of f, if for every $\overline{x}$ in $\mathcal{E}$ over x there exists a cocartesian lift of f with source $\overline{x}$.
- p is called a *cocartesian fibration*, if p has cocartesian lifts of every morphism.
- p is called an *isofibration*, if p has cocartesian lifts of all isomorphisms.

Remark 2.10. Cocartesian fibrations are classically called *Grothendieck opfibrations.* The Grothendieck construction provides an equivalence between cocartesian fibrations over $\mathcal{B}$ and pseudofunctors $\mathcal{B} \longrightarrow \mathbf{Cat}$. This is the 1-categorical version of Lurie's Straightening equivalence.

Remark 2.11. *Cocartesian lifts define pushforwards.* Let $p\colon \mathcal{E} \longrightarrow \mathcal{B}$ be a functor and suppose $f\colon x \longrightarrow y$ in $\mathcal{B}$ has cocartesian lifts. The fibers of p are (strict 1-) pullbacks in **Cat**.

$$\begin{tikzcd}
\mathcal{E}_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "p"] \\
\mathbf{1} \arrow[r, "x"] & \mathcal{B}
\end{tikzcd}$$

Up to iso, the category $\mathcal{E}_x$ has objects $e \in \mathcal{E}$ with $p(e) = x$ and morphisms $f\colon e \longrightarrow e'$ with $p(f) = \mathrm{id}_x$.

Now fix a choice of cocartesian lifts $\overline{f}_e\colon e \longrightarrow f_!(e)$ for every $e \in \mathcal{E}_x$. Let $\phi\colon a \longrightarrow b$ be in $\mathcal{E}_x$. Since $p(\overline{f}_b\phi) = f = \mathrm{id}_y\, f$ there exists a unique lift of id_y that produces the following commutative square in $\mathcal{E}$. We define $f_!(\phi) := \overline{\mathrm{id}_y}$.

$$\begin{tikzcd}
a \arrow[r, "\overline{f}_a"] \arrow[d, "\phi"'] & f_!(a) \arrow[d, dashed, "\exists!\ \overline{\mathrm{id}_y}"] \\
b \arrow[r, "\overline{f}_b"] & f_!(b)
\end{tikzcd}$$

$$\begin{tikzcd}
x \arrow[rr, "f"] \arrow[dr, "p(\phi\overline{f}_b)"'] & & y \arrow[dl, "\mathrm{id}"] \\
& y &
\end{tikzcd}$$

Moreoever, the choice of lifts we had to make was not essential: any two pushforward functors along f obtained this way are uniquely isomorphic. (This follows from the universal property of cocartesian lifts.)

Observation 2.12. How can we recover the data of $\mathcal{O}$ from $p\colon \mathcal{O}^{\otimes} \longrightarrow \mathbb{F}_*$?

- Objects of $\mathcal{O}$ are in the fiber $\mathcal{O}^{\otimes}_{\langle 1\rangle}$.
- Lifts of the active map $\alpha_n\colon \langle n\rangle \longrightarrow \langle 1\rangle$ are the n-ary operations.

- Composition in $\mathcal{O}^\otimes$ encodes composition and symmetries in $\mathcal{O}$.

Observation 2.13.

1. *All inert maps have cocartesian lifts.* Let $f\colon \langle n\rangle \longrightarrow \langle m\rangle$ be inert. Given any $(x_1,\dots,x_n) \in \mathcal{O}^\otimes_{\langle n\rangle}$ we need to construct

$$(x_1,\dots,x_n) \xrightarrow{(f,\Phi)} (y_1,\dots,y_m)$$

with $\Phi_j\colon (x_{f^{-1}(j)}) \longrightarrow y_j$, so choose $y_j \coloneqq x_{f^{-1}(j)}$ and $\Phi_j = \mathrm{id}$. Now consider the following situation:

$$\begin{tikzcd}
\mathcal{O}^\otimes \arrow[ddd] & & (y_1,\dots,y_m) \arrow[dr, dashed, "(g,\Gamma)"] & \\
& (x_1,\dots,x_n) \arrow[ur, "(f,\Phi)"] \arrow[rr, bend right, "(h,\Psi)"'] & & (z_1,\dots,z_k) \\
& & \langle m\rangle \arrow[dr, "g"] & \\
\mathbb{F}_* & \langle n\rangle \arrow[ur, "f"] \arrow[rr, bend right, "h"'] & & \langle k\rangle
\end{tikzcd}$$

We need to show that (g,Γ) is uniquely determined. The bottom diagram yields $h = gf$ and by definition of composition in $\mathcal{O}^\otimes$ we require

$$\Psi_j \overset{!}{=} \Gamma_j \circ (\Phi_i)_{i\in g^{-1}(j)} = \Gamma_j \circ (\Phi_{f(i)})_{i\in h^{-1}(j)} = \Gamma_j.$$

Hence, the factorization is unique.

2. *For all $X \in \mathcal{O}^\otimes_{\langle n\rangle}$, $Y \in \mathcal{O}^\otimes_{\langle m\rangle}$ and cocartesian lifts $X \longrightarrow x_i$ of the Segal maps, we have a pullback:*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{O}^\otimes}(Y,X) \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \prod_{i=1}^n \mathrm{Hom}_{\mathcal{O}^\otimes}(Y,x_i) \arrow[d, "\prod_i p"] \\
\mathrm{Hom}_{\mathbb{F}_*}(\langle m\rangle,\langle n\rangle) \arrow[r] & \prod_{i=1}^n \mathrm{Hom}_{\mathbb{F}_*}(\langle m\rangle,\langle 1\rangle)
\end{tikzcd}$$

Consider the cocartesian lifts $X := (x_1, \dots, x_n) \longrightarrow x_j$ of the ρ_j from part 1. Let $\overline{f}\colon Y \longrightarrow X$ be a lift of $f\colon \langle m \rangle \longrightarrow \langle n \rangle$. The composite $Y \longrightarrow X \longrightarrow x_j$ specifies a single $\Phi_1\colon (y_i)_{i \in f^{-1}(j)} \longrightarrow x_j$. These recover exactly $\overline{f}$. If $X \longrightarrow \hat{x}_j$ is another cocartesian lift of ρ_i, then its related to the one from part 1 by a unique iso $\hat{x}_j \xrightarrow{\sim} x_j$.

3. *For all $x_1, \dots, x_n$ in $\mathcal{O}^{\otimes}_{\langle 1 \rangle}$ there exists an $X \in \mathcal{O}^{\otimes}_{\langle n \rangle}$ and cocartesian lifts of the Segal maps $X \longrightarrow x_i$.*

Remark 2.14. The combination of (2) and (3) is also called the "Segal condition". It implies $\mathcal{O}^{\otimes}_{\langle n \rangle} \simeq \prod_{i=1}^{n} \mathcal{O}^{\otimes}_{\langle 1 \rangle}$, i.e. the fiber over $\langle n \rangle$ looks like tuples.

It turns out that these conditions fully characterize $\mathcal{O}^{\otimes} \longrightarrow \mathbb{F}_*$.

Theorem 2.15. *An isofibration $p\colon \mathcal{E} \longrightarrow \mathbb{F}_*$ is equivalent to a category of operators $\mathcal{O}^{\otimes} \longrightarrow \mathbb{F}_*$, if and only if it satisfies conditions (1) - (3) of Observation 2.13.*

Proof. [Hau23, Proposition 2.2.11] □

Proposition 2.16. *Let $\mathcal{O}$ and $\mathcal{P}$ be operads. A functor*

$$\begin{tikzcd} \mathcal{O}^{\otimes} \arrow[rr, "F"] \arrow[dr] & & \mathcal{P}^{\otimes} \arrow[dl] \\ & \mathbb{F}_* & \end{tikzcd}$$

comes from a functor of operads if and only if F preserves cocartesian lifts of inert morphisms.

Proof. Exercise. *Hint: Preserving cocartesian lifts of inert morphisms should correspond to $F(x_1, \dots, x_n)$ being the same as $(Fx_1, \dots, Fx_n)$.* □

Proposition 2.17. *The following are equivalent for an isofibration $p\colon \mathcal{E} \longrightarrow \mathbb{F}_*$.*

- *$\mathcal{E} \simeq \mathcal{V}^{\otimes}_{\mathrm{opd}}$ (over $\mathbb{F}_*$) for a symmetric monoidal category $\mathcal{V}$.*
- *p is a cocartesian fibration and satisfies (1) - (3) of Observation 2.13.*
- *p is a cocartesian fibration and the functors*

$$\mathcal{E}_{\langle n \rangle} \xrightarrow{(\rho_{1,!}, \dots, \rho_{n,!})} \prod_{i=1}^{n} \mathcal{E}_{\langle 1 \rangle}$$

are equivalences for all $n \in \mathbb{N}$.

*Moreoever, a functor between such cocartesian fibrations corresponds to a strong monoidal functor if and only if if preserves **all** cocartesian morphisms.*

Upshot: We can identify **Opd** with the subcategory of $\mathbf{Cat}_{/\mathbb{F}_*}$ whose objects are isofibrations satisfying (1) - (3) in Observation 2.13 and morphisms are those that preserve cocartesian lifts of inert maps.

Remark 2.18. We can take this as a new definition of operads. Specifying fibrations is often easier in practice. In particular, this new definition lifts readily to the world of ∞-categories.

References

[Hau23] Rune Haugseng. "An allegedly somewhat friendly introduction to ∞-operads". Available at https://runegha.folk.ntnu.no/iopd.pdf. Mar. 2023.